Structurable CBERs and $\mathcal{L}_{\omega_1\omega}$-interpretations

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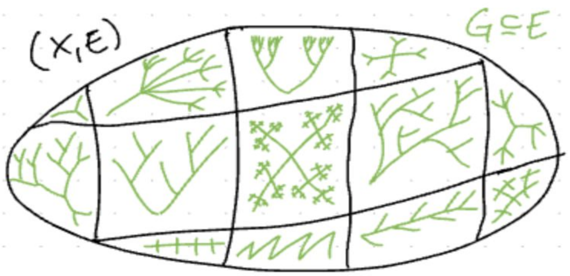
Example 1. E is treeable if there is a Borel way of putting a tree (a connected acyclic graph) on each E-class.



- Borel acyclic graph $G \subseteq E$
- On each E-class $C \in X/E$, $G \mid C^2$ is connected

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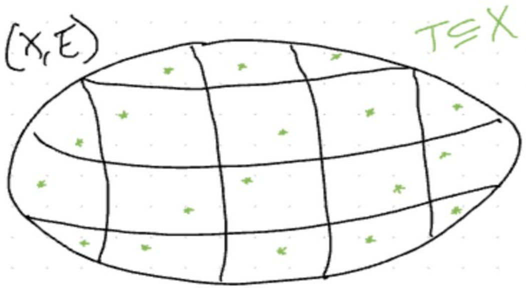
Example 2. E is smooth if there is a Borel way of picking a distinguished point in each E-class.



- Borel subset $T \subseteq X$
- On each E-class $C \in X/E$,
  $T|C$ is a singleton

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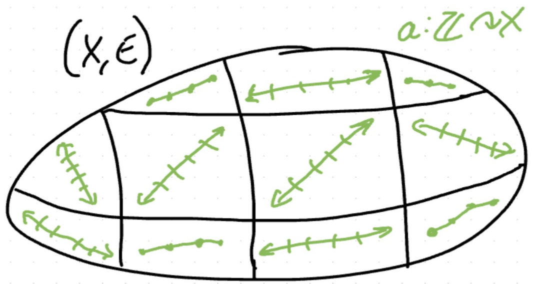
Example 3. E is hyperfinite if there is a Borel way of putting a transitive $\mathbb{Z}$ action on each E-class.



- Borel action $a\colon \mathbb{Z} \times X \to X$
- On each E-class $C \in X/E$,
  $a \mid (\mathbb{Z} \times C)$ is transitive

Structuring CBERs

Theme: global Borel structure that locally restricts to models of some theory $\mathscr{T}$.

Class of CBERs	Global structure	Local theory
Treeable	Borel graph $G \subseteq E$	$\mathscr{L}_{tree}$: binary relation symbol G $\mathscr{T}_{tree}$: $\forall x \neg xGx$ $\forall x \forall y\ (xGy \rightarrow yGx)$ $\forall x \forall y\ (x \neq y \leftrightarrow \exists n \in \mathbb{N}\ \exists z_1 \ldots \exists z_n\ (xGz_1 \ldots z_nGy))$
Smooth	Borel subset $T \subseteq X$	$\mathscr{L}_{smooth}$: unary relation symbol T $\mathscr{T}_{smooth}$: $\exists! x\ T(x)$
Hyperfinite	Borel action $a: \mathbb{Z} \times X \rightarrow X$	$\mathscr{L}_{hyp}$: unary function symbols a_n for each $n \in \mathbb{Z}$ $\mathscr{T}_{hyp}$: $\forall x\ \forall n,m \in \mathbb{Z},\ a_n(a_m(x)) = a_{n+m}(x)$ $\forall x\ a_1(x) = x$ $\forall x \forall y\ \exists n \in \mathbb{Z}\ a_n(x) = y$

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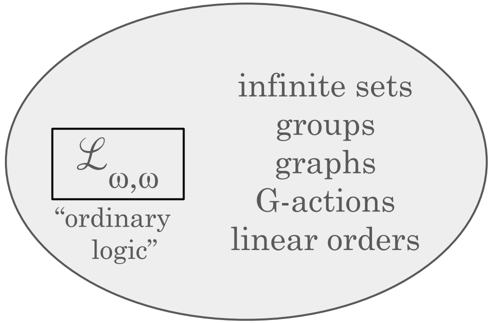
Expressiveness of various logics



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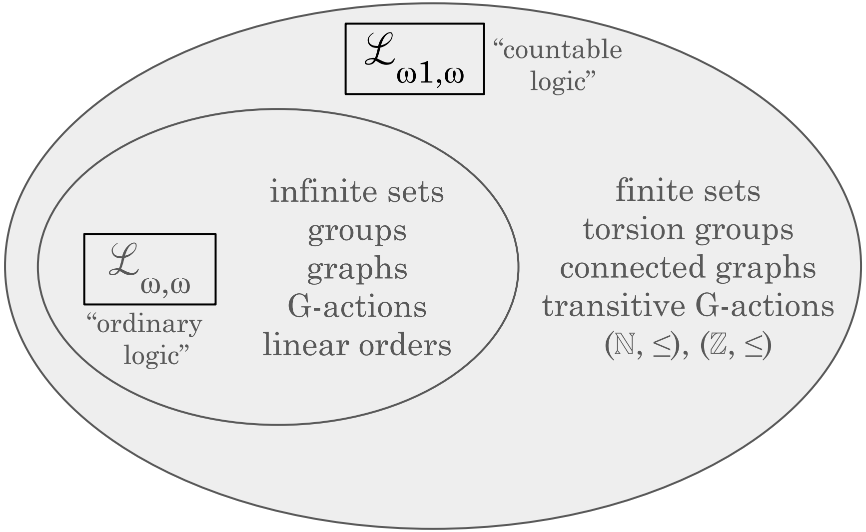

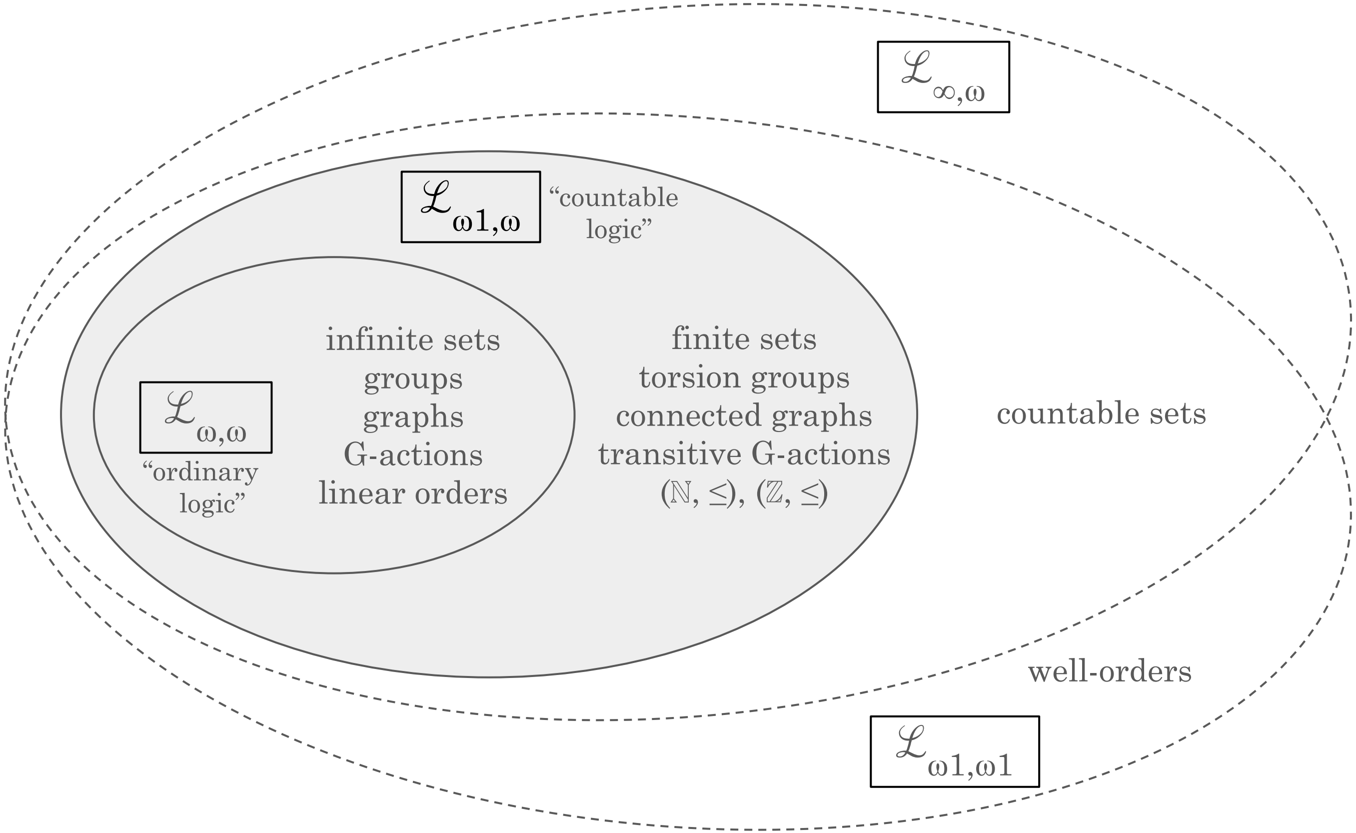

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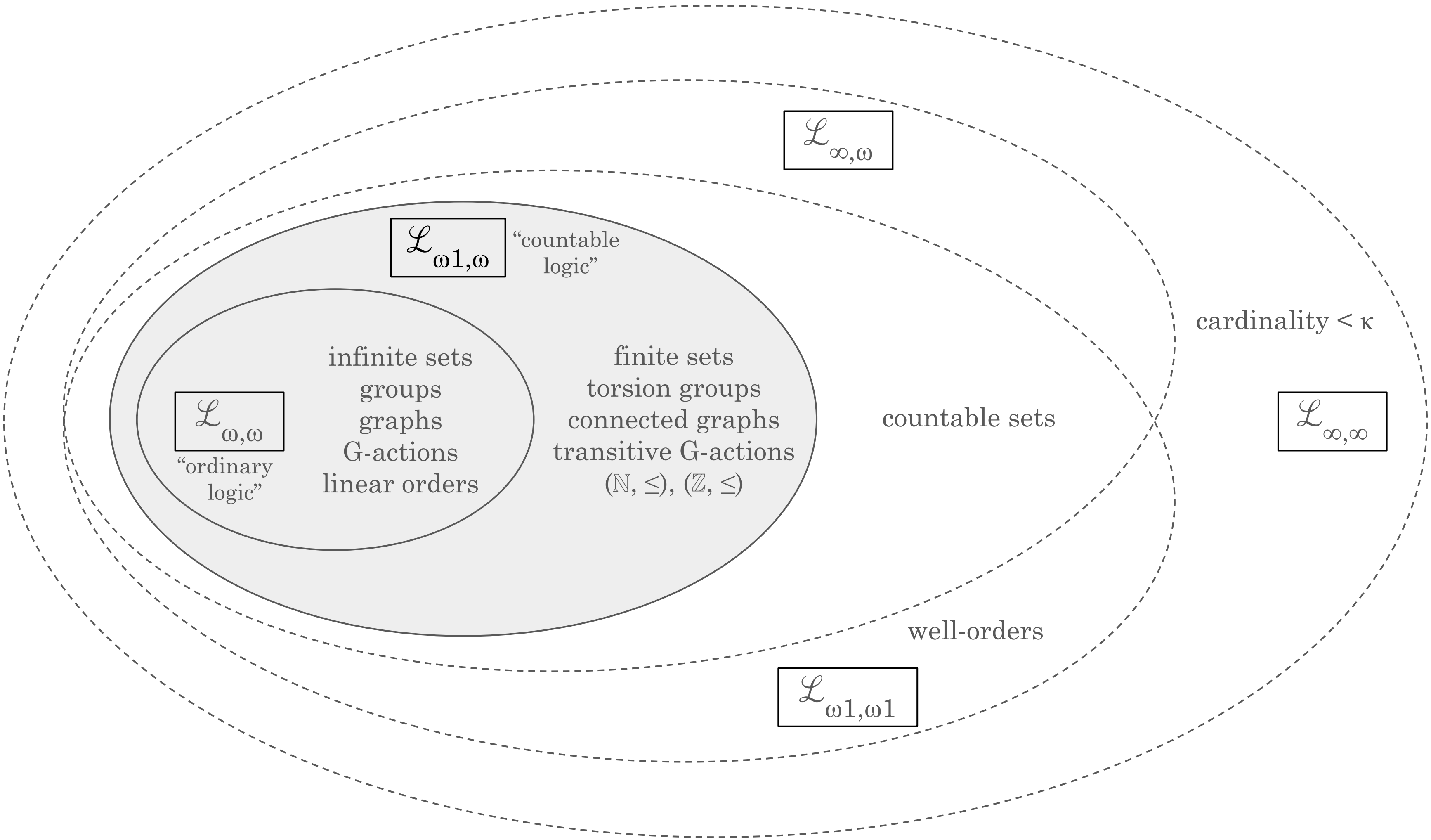

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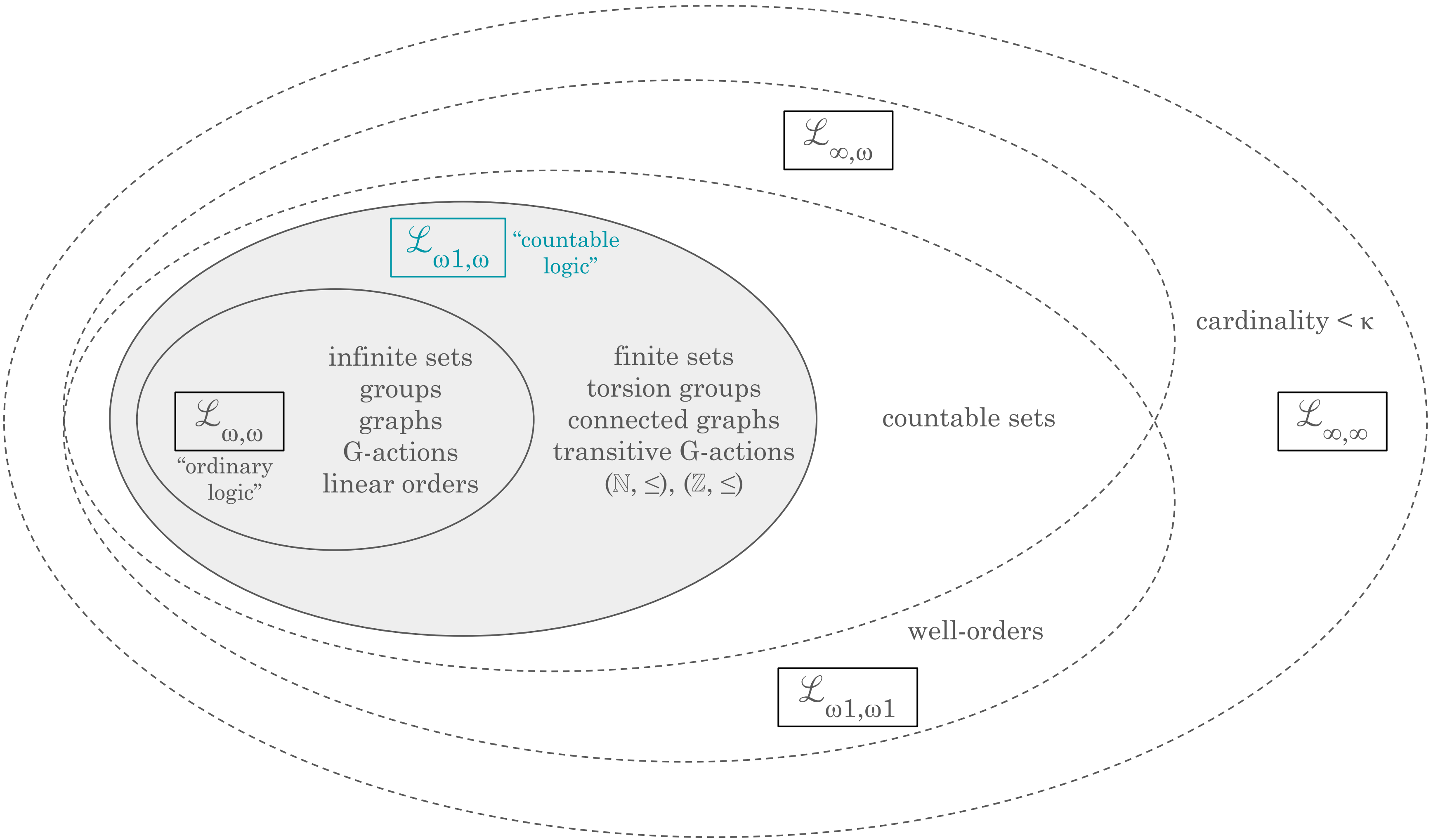

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Countable first order logic

(Relational) signature $\mathscr{L}$	set $\mathscr{L}$ of “relation symbols”, map arity: $\mathscr{L} \to \mathbb{N}$
($\mathscr{L}_{\omega 1,\omega}$-) formula $\varphi(\bar{x})$	built from symbols in $\mathscr{L}$ by applying negations, quantifiers, and **countable** conjunctions/disjunctions
($\mathscr{L}_{\omega 1,\omega}$-) theory $\mathscr{T}$	set of sentences: formulas without free variables
$\mathscr{L}$-structure $\mathscr{M} = (X, R^{\mathscr{M}})_{R \in \mathscr{L}}$	set X, interpretation $R^{\mathscr{M}} \subseteq X^{\text{arity}(R)}$ of each $R \in \mathscr{L}$

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We define the set of all $\mathscr{L}$-structures on X by

$$\mathrm{Mod}_X(\mathcal{L}) := \Pi_{R\in\mathcal{L}} 2^{X^{\mathrm{arity}(R)}}$$

And $\mathrm{Mod}_X(\mathscr{T})$ is the subset of $\mathrm{Mod}_X(\mathscr{L})$ consisting of just the models of $\mathscr{T}$ on X.

For countable X, $\mathscr{L}$, and $\mathscr{T}$, $\mathrm{Mod}_X(\mathscr{T})$ is a standard Borel space.

Structuring CBERs

Definition. Let E be a CBER on X, $\mathscr{L}$ a countable signature, $\mathcal{M}$ an $\mathscr{L}$-structure on X, and $\mathscr{T}$ a (countable $\mathscr{L}_{\omega_1\omega}$-) theory.

Then $\mathcal{M}$ is a <u>$\mathscr{T}$-structuring</u> of E if:

1. $R^{\mathcal{M}} \subseteq X^{\text{arity}(R)}$ is Borel for each $R \in \mathscr{L}$.
2. If $\bar{a} \in R^{\mathcal{M}}$, then $a_1 E \ldots E a_n$.
3. For every E-class C, $\mathcal{M} | C$ is a model of $\mathscr{T}$.

We write $\mathcal{M}$: E ⊨ $\mathscr{T}$ and say that E is <u>$\mathscr{T}$-structurable</u>.

Free structurings

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Example 1. X standard Borel $\Rightarrow$ X has a countable separating family of Borel subsets U_k. The U_k's still separate points when restricted to any E-class. So every CBER is structurable by:

$$\mathcal{L}_{sep} : \text{unary relation symbols } U_k \text{ for } k \in \mathbb{N}$$

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Example 2. By the Luzin-Novikov theorem, there are countably many Borel functions f_i: X $\rightarrow$ X whose graphs cover E. Restricting to any E-class, we get that every CBER is structurable by:

$$\mathcal{L}_{LN} : \text{unary function symbols } f_i \text{ for } i \in \mathbb{N}$$
$$\mathcal{T}_{LN} : \forall x \forall y \bigvee_{i \in \mathbb{N}} f_i(x) = y$$

The Scott theory of a CBER

Proposition. For any CBER E on X, there is an $\mathscr{L}_{sep}$-theory $\mathscr{T}_E$ (the "Scott theory" of E) such that $\mathscr{T}_E$-structurings are equivalent to class-bijective Borel homomorphisms into E.

Proof sketch:

Note that models $\mathscr{M}$ of $\mathscr{T}_{sep}$ are equivalent to injections $U^{\mathscr{M}}$ into $2^{\mathbb{N}}$, and assume X is a Borel subset of $2^{\mathbb{N}}$. Then for any Borel subset $B \subseteq (2^{\mathbb{N}})^n$, there is an n-ary quantifier-free $\mathscr{L}_{sep}$-formula $\psi_B(\bar{x})$ such that $\mathscr{M} \vDash \mathscr{T}_{sep} \cup \{\psi_B(\bar{a})\} \Leftrightarrow U^{\mathscr{M}}(\bar{a}) \in B$.

So define $\mathscr{T}_E := \mathscr{T}_{sep} \cup \{\forall x \forall y\, \psi_E(x,y),\ \forall x \bigwedge_{i\in\mathbb{N}} \exists y\, \psi_{f\_i}(x,y)\}$, where f_i are LN-functions for E. Then $\mathscr{T}_E$ models bijections into E-classes, so a $\mathscr{T}_E$-structuring of a CBER F is equivalent to a class-bijective Borel homomorphism $F \to E$.

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- To ensure $\mathscr{N} \vDash \mathscr{T}$, need to require that $\mathcal{T} \vDash \alpha(\mathscr{T})$, i.e. every model of $\mathcal{T}$ also models $\alpha(\varphi)$ for each $\varphi \in \mathscr{T}$.

Interpretations

Definition. Let $(\mathscr{L}, \mathscr{T})$ and $(\mathscr{L}', \mathscr{T}')$ be theories.

An <u>interpretation</u> α from $\mathscr{T}$ to $\mathscr{T}'$ is a map
$\alpha: \mathscr{L} \to \{\mathscr{L}'$ formulas$\}$ such that

1. For each $R \in \mathscr{L}$, arity(R) = arity$(\alpha(R))$
2. $\mathscr{T}' \vDash \alpha(\varphi)$ for each $\varphi \in \mathscr{T}$

We write $\alpha: \mathscr{T} \to \mathscr{T}'$ and say that $\mathscr{T}'$ <u>interprets</u> $\mathscr{T}$.

α induces $\alpha^*: \mathrm{Mod}_{\mathbb{N}}(\mathscr{T}') \to \mathrm{Mod}_{\mathbb{N}}(\mathscr{T})$, a Borel map
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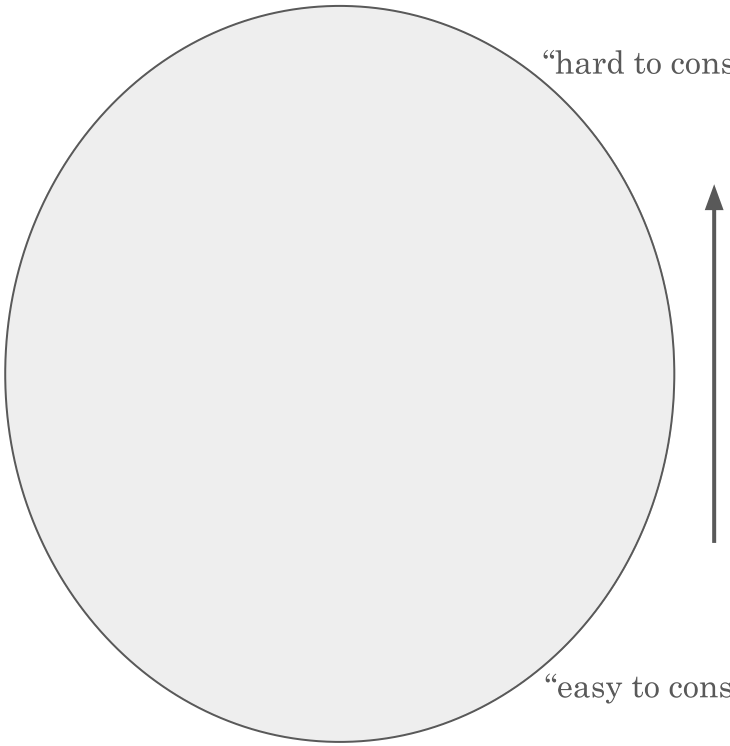
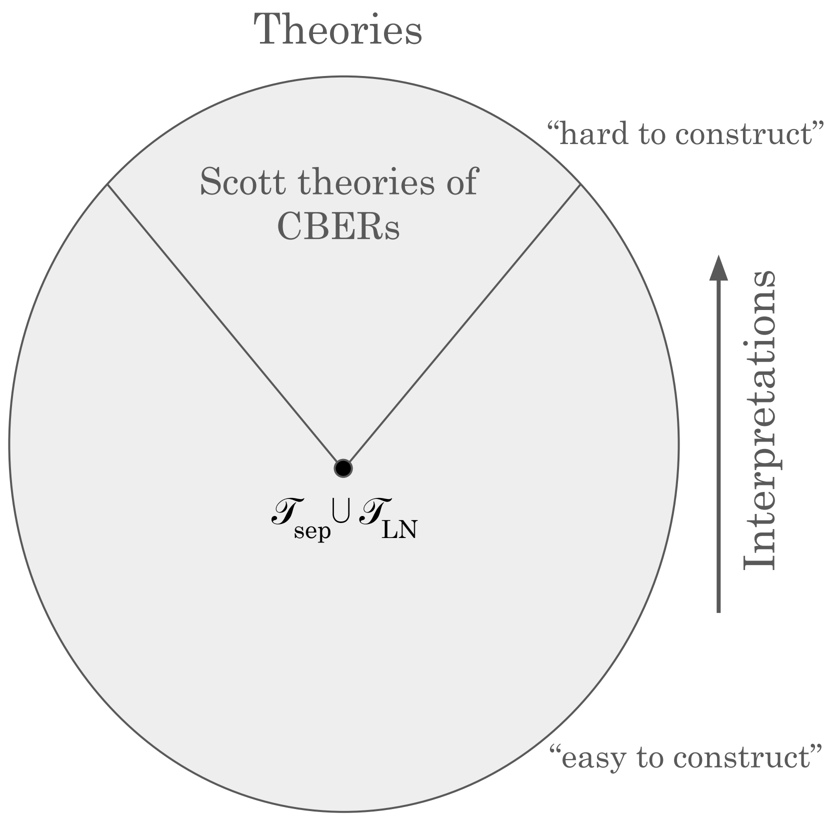


Interpretations

Example.

For any CBER E, the Scott theory $\mathscr{T}_E$ of E interprets $\mathscr{T}_{sep} \cup \mathscr{T}_{LN}$.

And every theory that interprets $\mathscr{T}_{sep} \cup \mathscr{T}_{LN}$ is a Scott theory $\mathscr{T}_E$ for some CBER E!



Feldman-Moore theorem as an interpretation

Theorem (FM): Let E be a CBER on X. Then E is the orbit equivalence relation of a Borel action of some countable group G on X.

Proof: Turn LN functions f_i for E into Borel involutions $g_i: X \to X$ whose graphs still cover E, and close under composition.

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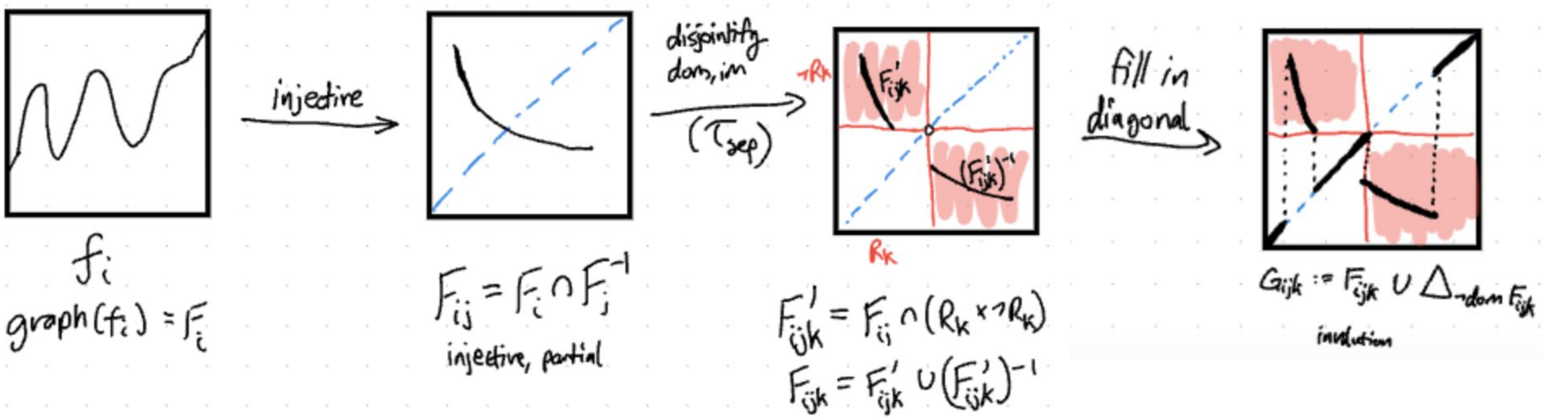
Carrying out this construction on classes, we get an interpretation $T_{FM} \to T_{sep} \cup T_{LN}$.



Interpretations and structurability

Write $\mathrm{Struc}_E(\mathscr{T})$ for the set of $\mathscr{T}$-structurings of E.

<u>Proposition</u>. Let E be any CBER and $\alpha: \mathscr{T} \to \mathscr{T}'$ an interpretation. Then α induces a map $\alpha^*: \mathrm{Struc}_E(\mathscr{T}') \to \mathrm{Struc}_E(\mathscr{T})$.

Proof: Let $\mathscr{M}$ be a $\mathscr{T}'$-structuring of E. Idea: apply α* classwise. So define an $\mathscr{T}$-structuring $\mathscr{N}$ of E by

$$\bar{a} \in R^{\mathscr{N}} :\Leftrightarrow a_1 E \ldots E a_n \;\&\; \bar{a} \in \alpha(R)^{\mathscr{M}}$$

Why is $R^{\mathscr{N}}$ Borel?

(α(R) may not be quantifier-free.)

Interpretations and structurability

Proof: (continued…)

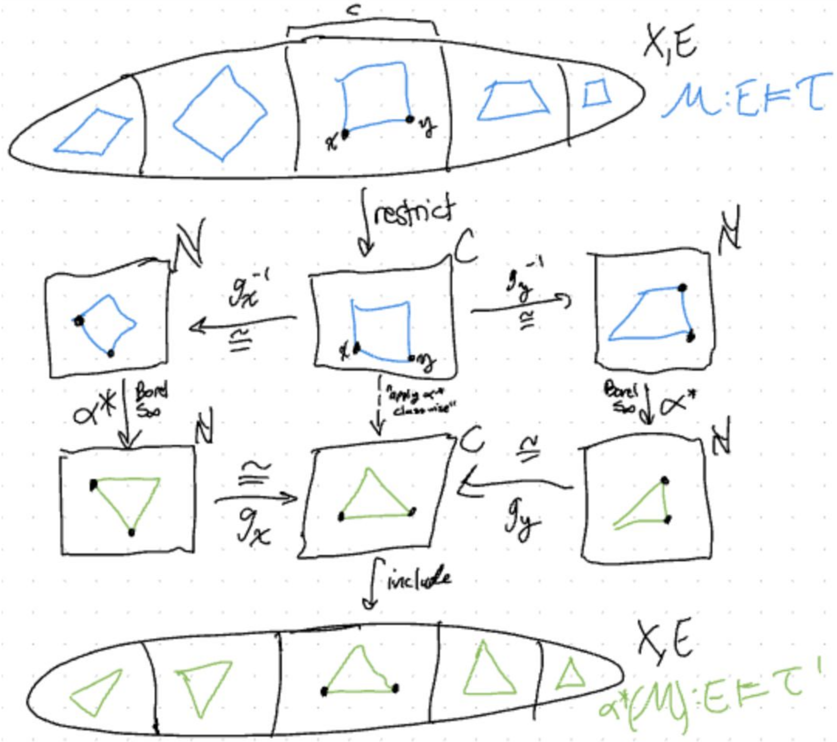
Luzin-Novikov

$f: \mathbb{N} \to X^X$

⇣

Borel $g: X \to X^{\mathbb{N}}$

$x \mapsto g_x \in \mathrm{Bij}(\mathbb{N}, [x]_E)$



Interpretations & class-bijective Borel homomorphisms

Theorem.

(class-bijective Borel homomorphisms between CBERs)

$\cong$

(interpretations between their Scott theories)

(Proof) An interpretation $\alpha: T_E \to \mathscr{T}_F$ induces a map $\alpha^*: \mathrm{Struc}_F(\mathscr{T}_F) \to \mathrm{Struc}_F(\mathscr{T}_E)$, and $\mathrm{Struc}_F(\mathscr{T}_E) \cong$ { class-bijective Borel homomorphisms $F \to E$ }, so letting $\mathrm{id}_F : F \models \mathscr{T}_F$ be the identity structuring of F, we get a class-bijective Borel homomorphism $\alpha^*(\mathrm{Id}_F)$: $F \to E$.

Conversely, given $f: F \to_B^{cb} E$, to get an interpretation $\mathscr{T}_E \to \mathscr{T}_F$, suffices to define $\alpha^*: \mathrm{Mod}(\mathscr{T}_F) \to \mathrm{Mod}(\mathscr{T}_E)$, or equivalently, α^*: {bijections to F-classes} $\to$ {bijections to E-classes}, which we obtain by precomposition ($g \mapsto f \circ g$).

CBERs and Theories